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NONNEGATIVELY CURVED LEAVES IN FOLIATIONS

S. R. ADAMS & A. FREIRE

Abstract

We use techniques from geometric analysis to prove that any Riemannian foliated measure space with finite total measure and leaves of nonnegative Ricci curvature has the property that a.e. leaf is the product of a compact Riemannian manifold and a flat Euclidean space

0. Introduction

We solve here a conjecture of R. Zimmer concerning nonnegatively curved leaves in foliations by Riemannian manifolds. We prove

Theorem 5.1. *Let $(M, \mathscr{F}, \mathscr{R})$ be a Riemannian foliated measure space with finite total measure such that a.e. leaf is complete and has nonnegative Ricci curvature. Then a.e. leaf can be written as the product of a compact Riemannian manifold and a flat Eucliean space.*

Thus, while it is possible to foliate a torus by lines, this theorem implies, for example, paraboloids cannot wrap around one another tightly enough to foliate a space of finite volume.

Theorem 5.1 is the counterpart of Zimmer's theorem [12] on nonpositively curved leaves in amenable foliations. In nonnegative curvature, it is not necessary to assume amenability; this comes for free since the leaves of the foliation have polynomial growth.

Our result, proved in the general context of foliated measure spaces, immediately implies the following theorem for foliations of Riemannian manifolds with holonomy-invariant measure (as defined for example in [8]):

Corollary 5.2. *Let $\mathscr{F}$ be a foliation of a compact manifold with a holonomy-invariant measure that is finite on compact subsets of transversals. Assume that almost every leaf (with respect to this measure) is a complete Riemannian manifold of nonnegative Ricci curvature. Then almost every*

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leaf of $\mathcal{F}$ is the Riemannian product of a compact manifold and a flat Euclidean space.

It is conceivable that, in the conclusion of Corollary 5.2, 'almost every leaf' may be replaced by 'every leaf.'

Theorem 5.1 is an analogue for foliations of a combination of two theorems. One is a theorem of S.-T. Yau [10, Theorem 7, p. 667] which states that a finite volume (complete) manifold of nonnegative Ricci curvature is compact. The other result [4, proof of Theorem 8.21, p. 150] states that the universal cover of a compact manifold of nonnegative Ricci curvature splits as compact cross Euclidean.

The idea in [10, Theorem 7, p. 667] is that the Busemann function of any ray in a nonnegatively curved manifold is Lipschitz and weakly superharmonic. Yau proves that such a function cannot occur in finite volume unless it is harmonic. In this case, by Lemma 3.2 below, any flow line of the gradient of the Busemann function would be an everywhere minimizing bi-infinite geodesic line. This line splits off (by [5] or [4, Theorem 8.17, p. 148]) as a direct factor, contradicting the finite volume hypothesis. Thus, the manifold cannot contain any rays and must therefore be compact.

We will attempt in this article to implement these ideas measurably on all of the leaves of a foliation. Unfortunately, it is sometimes not possible measurably to choose a ray from every leaf. The solution is to follow [12]: rather than choosing a single point in the boundary of each leaf, we instead use amenability to set up a system of measures on the boundaries of the leaves. As in [12], we can concentrate these measures on the anti-Euclidean boundaries, where none of the corresponding Busemann functions are harmonic. Then the superposition of these Busemann functions is not harmonic on any leaf. By an argument analogous to [10, Proposition 2, p. 667] (but done globally across the foliation), systems of leafwise Lipschitz subharmonic functions on a foliated measure space with finite total measure must be harmonic on almost every leaf (Theorem 4.6). Thus, we arrive at a contradiction. This implies that these anti-Euclidean boundaries must have been empty, which (by well-known facts from manifolds of nonnegative Ricci curvature) shows that the leaves are compact cross Euclidean.

To simplify matters, we do not actually use Busemann functions in this paper. Instead we use the abstract boundary of a locally compact space developed by Gromov ([7] or [2, §3, p. 21ff]).

Next we wish to describe two purely geometric applications of the main theorem (Corollary 5.3 and Theorem 6.5).

Recall that an action of a group Γ on a manifold M is said to be *properly discontinuous* if every $p \in M$ has an open neighborhood U such that $\{g \in \Gamma | gU \cap U \neq \emptyset\}$ is a finite subgroup of Γ.

Corollary 5.3. *Let N be a complete manifold of nonnegative Ricci curvature. Suppose N admits a properly discontinuous action by isometries of a discrete countable group Γ which has 'finite covolume' in the following sense: there exists an open subset $U \subset N$ with finite volume whose translates $\{\gamma U | \gamma \in \Gamma\}$ cover N. Then N splits isometrically as a product of a compact manifold and a flat Euclidean space.*

If the action is free, this is a corollary of the Cheeger-Gromoll splitting theorem [5, Theorem 3] and the above-mentioned result of Yau. The corollary follows from the foliated bundle construction in ergodic theory.

The second geometric application which we have in mind came out of a general suggestion of H. Furstenberg to try to understand conditions under which a Riemannian manifold might turn out to live inside a foliation of a finite measure space. As one example in that direction, we consider "almost periodic" metrics (Definition 6.1) on Euclidean d-space. In Lemma 6.4, we show that manifolds with such metrics can be "put into foliations," whereupon Theorem 5.1 has the following corollary.

Theorem 6.5. *Let g be a metric on $\mathbf{R}^d$ which is quasi-isometric to the usual flat metric, and is almost periodic to order two* (*Definition* 6.1). *Assume that either the sectional curvatures of g are all nonpositive or the Ricci curvatures of g are all nonnegative. Then g is flat.*

This is a generalization of the statement that the torus carries no nonflat metrics of nonpositive or nonnegative curvature. Our proof requires the full generality of foliations of measure spaces; the group A of Lemma 6.4 will typically not be a Lie group.

While the statement of Theorem 6.5 is completely geometric, we know of no proof of this result that does not proceed by embedding $\mathbf{R}^d$ into a foliation.

In §1, we set up basic definitions, including a definition of a foliation of a measure space. In §2, we recall a result from [1] on the existence of partitions of unity. In §3, we recall from [2] (or [7]) the abstract boundary of a Riemannian manifold. We study the analytic properties of the boundary function classes. We prove our analogue (Lemma 3.2) for [12, Lemma 5, p. 1016], one of the key ingredients in Zimmer's flat leaves theorem [12, Theorem 1, p. 1011]. In §4, we prove our foliation analogue (Theorem 4.6) of Yau's theorem on Lipschitz, weakly subharmonic functions on manifolds [10, Proposition 2, p. 667]. In §5, we prove our main result (Theorem 5.1) and Corollaries 5.2 and 5.3. In §6, we study almost periodic metrics and prove (Theorem 6.5) that such metrics, in the presence of curvature assumptions, are flat.

1. Basic definitions

All Borel spaces which we consider will be standard, i.e., Borel isomorphic to $[0, 1]$. A measure space will always be a standard Borel space with a σ-finite Borel measure.

For the remainder of the paper, we fix a positive integer n (to denote the dimension of the leaves of the foliation which we study). Let D denote the unit ball in $\mathbf{R}^n$.

If R is an equivalence relation on a set S, then, for all $s \in S$, we let $[s]_R$ denote the equivalence class of s. If $A \subseteq S$, then the *R-saturation of* A is denoted $[A]_R := \bigcup_{a \in A}[a]_R$. We say that R is *countable* if every equivalence class of R is countable.

Definition 1.1. If R is a countable equivalence relation on a Borel space B, then we say that R is *Borel* if its graph $R \subseteq B \times B$ is a Borel subset. A Borel automorphism $f\colon B \to B$ is called an *automorphism of* R if $(b, f(b)) \in R$ for all $b \in B$.

Definition 1.2. A countable Borel equivalence relation R on a finite measure space T is said to be *measure preserving* if every automorphism of R is measure preserving.

Definition 1.3. A *flow box* consists of

(i) a countable measure preserving Borel equivalence relation R on a finite measure space T; and

(ii) a measurable map $t \to g_t$ from T to the space of all Riemannian metrics on D.

Let μ_t be the volume density associated to g_t, and define a measure μ on $T \times D$ by

$$\mu(A) := \int_T \mu_t\{d \in D | (t, d) \in A\}\, dt$$

for all measurable $A \subseteq T \times D$. The resulting measure space is denoted $T \times_g D$.

In the next definition, no measurability assumption is made on the map $\mathscr{R}$. In fact, Definition 1.5 makes precise what it means for the map $\mathscr{R}$ to be "measurable."

Definition 1.4. Let $\mathscr{F}$ be a Borel equivalence relation on a measure space M. Let $\mathscr{R}\colon L \mapsto \mathscr{R}(L)$ be an association of an n-dimensional C^∞-Riemannian manifold structure to each equivalence class $L \subseteq M$. A *foliation chart for* $(M, \mathscr{F}, \mathscr{R})$ consists of

(i) a flow box $(T, R, t \mapsto g_t)$; and

(ii) a measure preserving Borel injection $u: T \times_g D \to M$ such that for all $d, d' \in D$ and all $t, t' \in T$,

$$(t, t') \in R \Leftrightarrow (u(t, d), u(t', d')) \in \mathscr{F},$$

and such that, for every equivalence class $L \subseteq M$, $u(T \times \{0\}) \cap L$ has no accumulation points in the topological space L.

Definition 1.5. Let $\mathscr{F}$ be a Borel equivalence relation on a finite measure space. Then $(M, \mathscr{F}, \mathscr{R})$ is a *Riemannian foliated measure space with finite total measure* if there exists a countable collection $\mathscr{A} := \{T_i, u_i: T_i \times_g D \to M\}_{i=1,2,\dots}$ of foliation charts such that

(i) $\bigcup_i u_i(T_i \times D)$ contains a.e. equivalence class; and

(ii) for every i and every $t \in T_i$, $u_i(t, \cdot)$ is an orientation preserving C^∞-isometry of (D, g_t) onto an open subset of the Riemannian manifold $[u_i(t, 0)]_{\mathscr{F}}$.

Such a collection $\mathscr{A}$ is called an *atlas*. Equivalence classes are called *leaves*.

Now, if f is a function on M, then it makes sense to speak of f being C^∞ along leaves. For such functions, we have a well-defined leafwise gradient ∇f and a leafwise Laplacian Δf.

2. Partitions of unity

Let $(M, \mathscr{F}, \mathscr{R})$ be a Riemannian foliated measure space with finite total measure.

Definition 2.1. If $\mathscr{A} = \{T_i, u_i: T_i \times_g D \to M\}$ is an atlas, then a sequence $f_i: M \to [0, 1]$ is called a *locally finite partition of unity subordinate to* $\mathscr{A}$ if

(i) $\sum_i f_i = 1$ a.e. on M;

(ii) each f_i is measurable on M and C^∞ along leaves;

(iii) for all i, $f_i = 0$ on $M \backslash u_i(T_i \times D)$;

(iv) for all i and $t \in T_i$, $d \mapsto f_i(t, d): D \to \mathbf{R}$ has compact support; and

(v) if L is a leaf and $K \subseteq L$ is compact, then $f_i|K \equiv 0$ for all but finitely many i.

Lemma 2.2 [1, Lemma 2.2]. *There exists an atlas $\mathscr{A}$ for $(M, \mathscr{F}, \mathscr{R})$ which has a locally finite partition of unity subordinate to it.*

3. Boundary theory in Riemannian manifolds

Let L be an n-dimensional complete Riemannian manifold. A *line in* L is a bi-infinite (i.e., parameterized by $\mathbf{R}$) geodesic which is distance minimizing between any two of its points. A *ray in* L is a positive-time (i.e., parameterized by $[0, \infty])$ geodesic which minimizes the length of any segment. All bi-infinite geodesics, positive-time geodesics, rays, lines, and geodesic segments will be parameterized by arc length. Let $d: L \times L \to \mathbf{R}$ denote the distance function on L.

Following [2, §3, p. 21], we let $C(L)$ denote the space of continuous functions on L in the topology of uniform convergence on compact sets. We let $C_*(L)$ denote the quotient of $C(L)$ by the constant functions and give this space the quotient topology. An element of $C_*(L)$ will be called a *function class*. We embed L in $C_*(L)$ by identifying a point $l \in L$ with the function class of $d(\cdot, l)$ in $C_*(L)$. Then ∂L denotes the boundary of L in $C_*(L)$; it is a compact metric space.

Note that, if $h \in C_*(L)$, then it makes no sense to evaluate h at a point $l \in L$. However, it does make sense to compute the change in h between l and l'. That is, if $h^* \in C(L)$ is any preimage of h, then we define

$$h|_l^{l'} := h^*(l') - h^*(l).$$

We will denote the C^∞ functions of compact support by $C_C^\infty(L)$.

A function on L will be said to be *Lipschitz*-1 if, for all $l, l' \in L$, $|f(l') - f(l)| \leq d(l', l)$. This property is invariant across function classes, so it makes sense to speak of an element of $C_*(L)$ being *Lipschitz*-1. Note that every element of ∂L, being a limit of Lipschitz-1 function classes, is again a Lipschitz-1 function class.

Other concepts which descend to $C_*(L)$ are *differentiability, differentiability at a point* $l \in L$, *harmonicity, superharmonicity, weak superharmonicity*. (A function $h^* \in C(L)$ is *weakly superharmonic* if $\int h^* \Delta g \leq 0$ for all $g \in C_C^\infty(L)$ satisfying $g \geq 0$ on L. To see that this concept is invariant across function classes, note that Stokes' theorem implies that $\int \Delta g = 0$ for all $g \in C_C^\infty(L)$.) At a point $l \in L$ of differentiability of $h \in C_*(L)$, the gradient $\nabla h(l) \in T_l L$ is well defined.

Lemma 3.1. *If $h \in \partial L$ and h is differentiable at $l \in L$, then* $|\nabla h(l)| = 1$.

Proof. Let $l_1, l_2, \cdots \in L$ be a sequence such that $d(\cdot, l_i) \to h$ mod constant functions. Let g_i be a length minimizing geodesic segment from l to l_i. Passing to a subsequence, $g_i'(0)$ converges. Let g be the positive-time geodesic such that $g_i'(0) \to g'(0)$. Then, since the length of g_i tends toward infinity as $i \to \infty$, we see that $g\colon [0, \infty) \to L$ is a ray.

Claim. For all $s > 0$, $h|_{g(0)}^{g(s)} = -s$. Discarding finitely many i, we may assume $d(l, l_i) > s$, for all i. Thus, for all i,

$$d(l, g_i(s)) + d(g_i(s), l_i) = d(l, l_i). \tag{1}$$

Let $\varepsilon > 0$ be arbitrary. Since $d(\cdot, l_i) \to h$ mod constants, we may assume that, for all i,

$$|h|_{g(0)}^{g(s)} - d(\cdot, l_i)|_{g(0)}^{g(s)}| < \varepsilon. \tag{2}$$

Finally, discarding finitely many i we may assume that the angle between $g'(0)$ and $g_i'(0)$ is so small that, for all i,

$$d(g(s), g_i(s)) < \varepsilon. \tag{3}$$

By the triangle inequality,

$$0 \le d(l, g(s)) + d(g(s), l_i) - d(l, l_i).$$

On the other hand, by (1) and applying the triangle inequality twice,

$$\begin{aligned} &d(l, g(s)) + d(g(s), l_i) - d(l, l_i) \\ &\quad = [d(l, g(s)) - d(l, g_i(s))] + [d(g(s), l_i) - d(g_i(s), l_i)] \\ &\quad \le d(g(s), g_i(s)) + d(g(s), g_i(s)). \end{aligned}$$

So, by (3),

$$0 \le d(l, g(s)) + d(g(s), l_i) - d(l, l_i) < 2\varepsilon.$$

Since $d(l, g(s)) = s$ and $l = g(0)$, we have

$$0 \le s + d(\cdot, l_i)|_{g(0)}^{g(s)} < 2\varepsilon,$$

which and (2) give

$$|h|_{g(0)}^{g(s)} + s| < 3\varepsilon$$

for all $\varepsilon > 0$, proving the claim.

From the claim, it follows that $\nabla h(l) \cdot g'(0) = -1$, so that $|\nabla h(l)| \ge 1$. On the other hand, h is a limit of Lipschitz-1 function classes and is therefore Lipschitz-1, from which we have $|\nabla h(l)| \le 1$. q.e.d.

Lemma 3.2 below is our analogue to [12, Lemma 5, p. 1016]. Since harmonic functions are C^1, Lemma 3.2 shows that a manifold with a harmonic boundary function contains a line. By [5] or [4, Theorem 8.17, p. 148], this line splits off as a direct factor (assuming nonnegative Ricci curvature).

Lemma 3.2. *If $h \in \partial L$ and h is C^1, then L contains a line.*

Proof. Let $g: \mathbf{R} \to L$ be any flow line for ∇h. Then we claim that g is a line. Let $s, s' \in \mathbf{R}$, $s < s'$. We wish to show that $d(g(s), g(s')) = s' - s$.

By Lemma 3.1, $|\nabla h| \equiv 1$ on L, so g is parameterized by arc length. Thus

$$d(g(s), g(s')) \leq s' - s.$$

Let $q: [0, d] \to L$ be a length minimizing geodesic segment such that $q(0) = g(s)$ and $q(d) = g(s')$. Then

$$\begin{aligned} s' - s &= h(g(s')) - h(g(s)) = h(q(d)) - h(q(0)) \\ &= \int_0^d \nabla h(q(t)) \cdot q'(t)\, dt. \end{aligned}$$

Taking absolute values and using Lemma 3.1, we obtain

$$s' - s \leq \int_0^d |\nabla h(q(t))| \cdot |q'(t)|\, dt = \int_0^d 1 \cdot 1\, dt = d = d(g(s), g(s')).$$

q.e.d.

Next we prove that elements of ∂L are weakly superharmonic. This is similar to the proof in [10, p. 669, 1.10] that Busemann functions are weakly superharmonic; however, we follow a simpler notation. Before starting the proof, we make the following simple observation.

Lemma 3.3. *Let h be Lipschitz on L and let $\phi \in C_C^\infty(L)$. Let Q denote the set of points where h is differentiable. Then*

$$\int_L h\Delta\phi = -\int_Q \nabla h \cdot \nabla \phi.$$

Corollary 3.4. *Fix $l \in L$. Let $h := d(\cdot, l)$ and let C denote the cut locus of l. Then, for all $g \in C_C^\infty(L)$,*

$$\int_{L \setminus C} \nabla h \cdot \nabla g = -\int_L h\Delta g.$$

Recall that if $h \in C_*(L)$ and $g \in C_C^\infty(L)$, then $\int_L h\Delta g$ is well defined by Stokes' theorem.

Lemma 3.5. *Assume all Ricci curvatures of L are nonnegative. Then, for all $h \in \partial L$, h is Lipschitz-1 and weakly superharmonic.*

Proof (cf. [10, p. 669, 1.10] *and* [5, p. 123, 1.7]). Since every $d(\cdot, l)$, $l \in L$, is Lipschitz-1, it follows that every element of ∂L is as well.

Let $g \in C_C^\infty(L)$ satisfy $g \geq 0$ on L. We wish to show that $\int_L h\Delta g \leq 0$. Choose $l_1, l_2, \cdots \in L$ such that $h_i := d(\cdot, l_i) \to h$ mod constant functions. Let C_i denote the cut locus of l_i.

Fix a positive integer i and let $l := l_i$. Let S denote the unit sphere of T_lL. For all $v \in S$, let $c(v)$ denote the (possibly infinite) supremum of

$$\{t \in (0, \infty) | d(l, \exp_l(tv)) = t\}.$$

By truncating and convolving, choose a sequence of C^∞ functions $c_j: S \to (0, \infty)$ such that $c_j(v) < c(v)$ and $c_j(v) \to c(v)$ for all $v \in S$. Let $T_i^j := \{tv | v \in S,\ 0 \le t \le c_j(v)\}$ for $j = 1, 2, \cdots$. Then $\exp_l$ is injective on T_i^j and $\bigcup_j \exp_l(T_i^j) = L \backslash C_i$. Furthermore, if V is the radial, unit length, outward pointing vector field on $T_lL\backslash\{0\}$, then V points outward along the boundary of T_i^j. Note that, for any i, j and any $v \in T_i^j$, $d\exp(V_v) = \nabla h_i(\exp(v))$. Consequently, picking j large enough (depending on i), and setting $D_i := \exp_l(T_i^j)$, we may assume

(i) $\partial h_i/\partial \vec{n} \ge 0$ on ∂D_i, where $\vec{n}$ is the outward pointing normal to D_i;

(ii) $|\int_{L\backslash(D_i \cup C_i)} \nabla h_i \cdot \nabla g| < 1/i$; and

(iii) $|\int_{L\backslash(D_i \cup C_i)} g \Delta h_i| < 1/i$.

By Stokes' theorem,

$$0 \le \int_{\partial D_i} g \frac{\partial h_i}{\partial \vec{n}} = \int_{D_i} \nabla h_i \cdot \nabla g + \int_{D_i} g \Delta h_i.$$

Then, by (ii) and (iii),

$$\frac{-2}{i} \le \int_{L\backslash C_i} \nabla h_i \cdot \nabla g + \int_{L\backslash C_i} g \Delta h_i.$$

By Corollary 3.4,

$$\int_{L\backslash C_i} \nabla h_i \cdot \nabla g = -\int_L h_i \Delta g,$$

so

$$-\frac{2}{i} \le -\int_L h_i \Delta g + \int_{L\backslash C_i} g \Delta h_i.$$

Let $n := \dim(L)$. Since $h_i = d(\cdot, l_i)$, and the Ricci curvature is nonnegative, it follows from comparison theory that $\Delta h_i \le (n-1)/h_i$. Further, there are constants c_i such that $h_i - c_i \to h$ uniformly on compact sets as $i \to \infty$. By Stokes' theorem, $\int c_i \Delta g = 0$. We now have

$$-\frac{2}{i} \le -\int_L (h_i - c_i) a \Delta g + (n-1) \int_{L\backslash C_i} \frac{g}{h_i}.$$

Let $i \to \infty$ in this inequality. Since the l_i's leave compact sets, it follows that $h_i \to \infty$ uniformly on compact sets. We therefore have $0 \le -\int h\Delta g + 0$ or $\int h\Delta g \le 0$, as desired.

4. Systems of weakly subharmonic functions

Let L be a Riemannian manifold. A function $h \in C(L)$ is said to be *weakly subharmonic* if $\int h\Delta g \ge 0$ for all $g \in C_C^\infty(L)$ satisfying $g \ge 0$.

Let $(M, \mathscr{F}, \mathscr{R})$ be a Riemannian foliated measure space with finite total measure.

Definition 4.1. A *discrete section of* $(M, \mathscr{F}, \mathscr{R})$ is a set $S \subseteq M$ such that

(i) $[S]_{\mathscr{F}}$ has positive measure; and

(ii) for all $m \in M$, $S \cap [m]_{\mathscr{F}}$ has no accumulation points in the Riemannian manifold $[m]_{\mathscr{F}}$. (We allow $S \cap [m]_{\mathscr{F}}$ to be empty for some values of m.)

A *compatible system for* S is a collection, $\{h_s : [s]_{\mathscr{F}} \to \mathbf{R}\}_{s\in S}$, of weakly subharmonic functions satisfying

(i) $(s, s') \in \mathscr{F}|S$ implies $h_{s'} - h_s$ is a constant function; and

(ii) $(s, m) \mapsto h_s(m) : (S \times M) \cap \mathscr{F} \to \mathbf{R}$ is measurable; the measure on $(S \times M) \cap \mathscr{F}$ is defined by

$$A \mapsto \int_M |A \cap (S \times \{m\})| \, d\mu(m),$$

where $|\cdot|$ denotes cardinality, and μ denotes the measure on M.

Definition 4.2. If $\psi : M \to \mathbf{R}$ is measurable and leafwise C^∞, then we say that ψ has *compact support* if there exists an atlas $\mathscr{A}$ and a locally finite partition of unity $\{f_i\}_i$ subordinate to $\mathscr{A}$ such that $f_i\psi \equiv 0$ for all but finitely many i. In this case, we say that $\mathscr{A}$ and $\{f_i\}_i$ are *adapted to* ψ.

Let $u : T \times_g D \to M$ be a foliation chart. Let S be a discrete section and let $\{h_s\}_{s\in S}$ be a compatible system. For all $t \in T$, let $h_t : D \to \mathbf{R}$ be defined by

$$h_t(d) := h_s(u(t, d)) - h_s(u(t, 0)),$$

where s is any element of $S \cap [t]_{\mathscr{F}}$. (The result is independent of s.) In the case where $S \cap [t]_{\mathscr{F}} = \varnothing$, we define $h_t \equiv 0$ on D. We call $\{h_t\}_{t\in T}$ the *transfer of* $\{h_s\}_{s\in S}$ *to* T.

Let $\{\zeta_t \in C_C^\infty(D)\}_{t\in T}$ be a system of functions which is *transversally measurable* in the sense that $(t, d) \mapsto \zeta_t(d) : T \times D \to \mathbf{R}$ is measurable.

Define $\zeta\colon M \to \mathbf{R}$ by $\zeta(u(t, d)) = \zeta_t(d)$ and by $\zeta \equiv 0$ on $M \backslash u(T \times D)$. We call ζ the *transfer of* $\{\zeta_t\}_{t\in T}$ *to* M.

Definition 4.3. Let $\mathscr{A} = \{T_i, u_i\colon T_i \times_g D \to M\}_i$ be an atlas, and $\{f_i\}_i$ a partition of unity adapted to a nonnegative valued, leafwise C^∞ function $\psi\colon M \to \mathbf{R}$ of compact support. Let S be a discrete section and let $h := \{h_s\}_{s\in S}$ be a compatible system of weakly subharmonic functions. For all i and all $t \in T_i$, define $\psi_t^i\colon D \to \mathbf{R}$ by

$$\psi_t^i(d) := f_i(u_i(t, d))\psi(u_i(t, d)).$$

We then define

$$\int_M h\Delta\psi := \sum_i \int_{T_i} \left[\int_{(D, g_t^i)} h_t^i \Delta\psi_t^i\right] dt,$$

where $\{h_t^i\}_t$ is the transfer of h to T_i.

This is independent of $\mathscr{A}$, $\{f_i\}_i$ adapted to ψ. The proof of this independence follows the same lines as the proof that integration on manifolds is well defined, so we omit the tedious details. However, three points are worth noting. The first is that it is necessary in the proof to justify interchanging $\sum_j$ with $\int_{T_i}$ and with $\int_{(D, g_t^i)}$. Interchanging with $\int_{(D, g_t^i)}$ follows from ordinary calculus on manifolds since the sums involved are finite and the functions are continuous with compact support. Interchanging with $\int_{T_i}$ follows from the general fact that if the α_j's are nonnegative measurable functions, then $\sum_j \int \alpha_j = \int \sum_j \alpha_j$. Thus, if the $\{h_s\}_{s\in S}$ were not weakly subharmonic, $\int h\Delta\psi$ might not be well defined.

The second part is that care is required, since these integrals cannot be transferred down to the measure space; the integrands may not be integrable functions in M. On overlaps, it is necessary to transfer the leafwise integrals across first; then use the holonomy invariance of the transverse measures.

The third point is that, by Stokes' theorem, for any i and any $t \in T_i$, $\int_{(D, g_t^i)} \Delta\psi_t^i = 0$, so that the value of $\int_{(D, g_t^i)} h_t^i \Delta\psi_t^i = 0$ is unaffected if we change h_t^i by a constant.

To set up the next lemma, let $u\colon T \times_g D \to M$ be a fixed foliation chart again, let $\{\zeta_t \in C_C^\infty(D)\}_{t\in T}$ be transversally measurable, and let $h := \{h_s\}_{s\in S}$ be a compatible system of weakly subharmonic functions for some discrete section S. Let $\zeta\colon M \to \mathbf{R}$ be the transfer of $\{\zeta_t\}_{t\in T}$ to M, and $\{h_t\}_{t\in T}$ the transfer of $\{h_s\}_{s\in S}$ to T.

Lemma 4.4. *Assume that* $\zeta_t \geq 0$ *for all* $t \in T$. *If there is a single compact subset of* D *containing the support of every* ζ_t, *then* ζ *has compact* *support and*

$$\int_M h\Delta\zeta = \int_T \left[\int_{(D,g_t)} h_t \Delta\zeta_t\right] dt. \tag{1}$$

Proof. Let $\{f_i\}_{i=1,2,\cdots}$ be subordinate to some atlas $\mathscr{A} = \{T_i, u_i\colon T_i \times_g D \to M\}_i$. Let $T_0 := T$ and $u_0 := u$.

Let $D' \subseteq D$ be a closed ball of radius < 1 about 0 containing the support of every ζ_t. Let $D'' \subseteq D$ be an open ball of radius < 1 about 0 containing D'. Let $s\colon D \to \mathbf{R}$ be a C^∞ function satisfying $s|D' \equiv 1$ and $s|(D\backslash D'') \equiv 0$.

Let $f_0^*\colon M \to \mathbf{R}$ be defined by $f_0^*(u_0(t, d)) = s(d)$ and by $f_0^* \equiv 0$ on $M\backslash u_0(T \times D)$. For $i = 1, 2, \cdots$, define $f_i^*\colon M \to \mathbf{R}$ by $f_i^*(m) = f_i(m)(1 - f_0^*(m))$.

Now we see that $\{f_i^*\}_{i=0,1,\cdots}$ is subordinate to the atlas $\{T_i, u_i\colon T_i \times_g D \to M\}_{i=0,1,\cdots}$ and that $f_i^*\zeta = 0$ for all $i > 0$. Thus, ζ has compact support and

$$\int_M h\Delta\zeta = \int_{T_0} \left[\int_{(D,g_t^0)} h_t^0 \Delta\zeta_t\right] dt,$$

where $\{h_t^0\}_{t\in T}$ is the transfer of $\{h_s\}_{s\in S}$ to T_0. This is the same as equation (1).

Lemma 4.5. *Let $\psi\colon M \to [0, \infty)$ be leafwise C^∞ with compact support, and let h be a compatible system of weakly subharmonic functions over some discrete section. Then $\int_M h\Delta\psi \geq 0$.*

Proof. In the notation of Definition 4.3, for all i and all $t \in T_i$, h_t^i is weakly subharmonic on (D, g_t^i) and $\psi_t^i \geq 0$, and therefore:

$$\int_{(D,g_t^i)} h_t^i \Delta\psi_t^i \geq 0.$$

The result follows on integrating over T_i with respect to t and then summing over i. q.e.d.

If S is a discrete section, then a subset $S' \subseteq S$ will be said to be *null* (resp. *conull*) if $[S']_{\mathscr{F}}$ (resp. $[S\backslash S']_{\mathscr{F}}$) is of measure zero. Thus the phrase "a.e. $s \in S$" is well defined. The main theorem of this section is a foliation analogue of [10, Proposition 2, p. 667]:

Theorem 4.6. *Let $h := \{h_s\}_{s\in S}$ be a compatible system of Lipschitz-1, weakly subharmonic functions on $(M, \mathscr{F}, \mathscr{R})$. Then h_s is harmonic for a.e. $s \in S$.*

Proof. Assume for a contradiction not. Restricting S to a nonnull subset, we may assume that h_s is not harmonic for all $s \in S$.

Let $u\colon T \times_g D \to M$ be a foliation chart, and let $\{h_t \in C^\infty(D)\}_{t\in T}$ be the transfer of $\{h_s\}_{s\in S}$ to T. With a careful choice of T and u, no h_t will be harmonic. Then, by regularity theory, no h_t will be weakly harmonic. However, every h_t is weakly subharmonic and Lipschitz-1, by Lemma 3.5.

Let $C_C^\infty(D)^+$ denote the set of all nonnegative, compactly supported, C^∞ function on D in the usual C^∞ topology. Let $\zeta_1, \zeta_2, \cdots \in C_C^\infty(D)^+$ be a countable dense subset. For every $t \in T$, let k_t be the smallest positive integer such that

$$\int_D h_t \Delta \zeta_{k_t} > 0.$$

Let $\zeta_t := \zeta_{k_t}/\max \zeta_{k_t}$ for all $t \in T$. Restricting T, we may assume that there is a single compact subset $D' \subseteq D$ containing the support of every ζ_t.

Let m_T denote the measure on T. By definition of a flow box (Definition 1.3), $m_T(T) < \infty$. Let ζ be the transfer of $\{\zeta_t\}_{t\in T}$ to M. Then, by Lemma 4.4, ζ has compact support and $\int_M h\Delta\zeta \le \int_T 1\, dt = m_T(T) < \infty$.

Fix an atlas $\mathscr{A} := \{T_i, u_i\colon T_i \times_g D \to M\}_{i=0,1,\cdots}$ which a subordinate locally finite partition of unity $\{f_i\}_{i=0,1,\cdots}$ such that $T_0 = T$ and $u_0 = u$ and such that $f_0 \equiv 1$ on $u_0(T_0 \times D')$. This can be done as in the proof of Lemma 4.4.

For every $m \in M$, let d_m denote the distance function in the Riemannian manifold $[m]_{\mathscr{F}}$. If $m' \in [m]_{\mathscr{F}}$ and $S \subseteq M$, then we denote the infimum of distances from m' to points of $S \cap [m]_{\mathscr{F}}$ by $d_m(m', S)$, with the convention that this infimum equals ∞ if $S \cap [m]_{\mathscr{F}} = \varnothing$.

Let $\varepsilon > 0$ be arbitrary. Let $K, N \in \mathbf{R}$ be sufficiently large that there exists a subset $T' \subseteq T$ satisfying

(i) $\int_{u(T''\times D')} h\Delta\zeta < \varepsilon$, where $T'' := T \backslash T'$;

(ii) $f_i(m) = 0$ if $d_m(m, u(T' \times D')) < 4/\varepsilon$ and $i > N$; and

(iii) $|\nabla f_i(m)| < K$ if $d_m(m, u(T' \times D')) < 4/\varepsilon$ and $1 \le i \le N$.

Let $\phi\colon M \to \mathbf{R}$ be a measurable, leafwise C^∞ function satisfying

(iv) $\phi(m) = 0$ if $d(m, u(T' \times D')) \ge 4/\varepsilon$;

(v) $\phi \equiv 1$ on $u(T' \times D')$; and

(vi) $|\nabla\phi| < \varepsilon$ on M.

This can be done by making ϕ a smoothing of a gradually decaying function of the leafwise distance from the set $u(T' \times D')$ (see [1, proof of Lemma 4.2]). By (ii) and (iv), ϕ has compact support, and $\mathscr{A}$, $\{f_i\}_i$ are adapted to ϕ.

Let $\zeta' \colon M \to \mathbf{R}$ be the transfer of $\{\zeta_t\}_{t\in T'}$ to M. Then, by (v), $\phi \geq \zeta'$ on M. So, by Lemma 4.5, $\int_M h\Delta(\phi - \zeta') \geq 0$. Thus, by (i),

$$\left[\int_M h\Delta\zeta\right] - \varepsilon < \int_M h\Delta\zeta' \leq \int_M h\Delta\phi.$$

Let $f_t^i(d) := f_i(u_i(t, d))$, $\tilde{\phi}_t^i(d) := \phi(u_i(t, d))$, and $\phi_t^i := (\tilde{\phi}_t^i) \cdot (f_t^i)$. Let $\{h_t^i\}_{t\in T_i}$ be the transfer of $\{h_s\}_{s\in S}$ to T_i. Then, by Lemma 3.3 and Definition 4.3,

$$\begin{aligned}\int_M h\Delta\phi &= \sum_i \int_{T_i} \int_{(D, g_t^i)} h_t^i \Delta\phi_t^i \\ &= -\sum_i \int_{T_i} \int_{Q_t^i} \nabla h_t^i \cdot [(\nabla\tilde{\phi}_t^i)\cdot(f_t^i) + (\tilde{\phi}_t^i)\cdot(\nabla f_t^i)]\, dt,\end{aligned}$$

where $Q_t^i \subseteq D$ is the set of points where h_t^i is differentiable. Let Q denote the set of points of leafwise differentiability of h. Then ∇h is a well-defined vector field on Q, and we may transfer the integrals down to $(M, \mathscr{F}, \mathscr{R})$. Since everything in sight is bounded (including $|\nabla h| \mid Q$, since every h_s is Lipschitz-1), and since $\sum_i f_i \equiv 1$ and $\sum_i \nabla f_i = 0$ on M, we may interchange summation and integration and obtain

$$\begin{aligned}\int_M h\Delta\phi &= -\int_Q \nabla h \cdot \left[(\nabla\phi)\cdot\left(\sum_i f_i\right) + (\phi)\cdot\left(\sum_i \nabla f_i\right)\right] \\ &\leq \int_Q |\nabla h|\cdot|\nabla\phi| \leq \int_Q 1\cdot\varepsilon = \varepsilon\mu(Q).\end{aligned}$$

Thus

$$\left[\int_M h\Delta\zeta\right] - \varepsilon < \varepsilon\mu(Q)$$

for all $\varepsilon > 0$. Since $\int_M h\Delta\zeta > 0$ and $\mu(Q) = \mu(M) < \infty$, this is a contradiction.

5. The main theorem

By [5] or [4, Theorem 8.17, p. 148], any complete Riemannian manifold L of nonnegative Ricci curvature can be written $L = L_1 \times L_2$, where L_1 contains no lines, and L_2 is a flat Euclidean space. Let $\partial^1 L$ be the image of ∂L_1 under the map $\partial L_1 \to \partial L$ induced from the projection map $L \to L_1$. Then $\partial^1 L$ is a closed subset of ∂L. We call $\partial^1 L$ the *anti-Euclidean boundary of* L. By Lemma 3.2, no element of $\partial^1 L$ is C^1.

For any compact metric space K, we denote the space of probability measures on K by $M(K)$; it is given the weak-* topology.

We can now prove the main theorem of this paper.

Theorem 5.1. *Let $(M, \mathscr{F}, \mathscr{R})$ be a Riemannian foliated measure space with finite total measure such that a.e. leaf is complete and has nonnegative Ricci curvature. Then a.e. leaf can be written as the product of a compact Riemannian manifold and a flat Euclidean space.*

Proof. Assume for a contradiction not. Then there exists a discrete section $S \subseteq M$ such that $\partial^1[s]_{\mathscr{F}} \neq \emptyset$ for all $s \in S$. Let T_s be the tangent space of the Riemannian manifold $[s]_{\mathscr{F}}$ at the point s. Let K_s denote the image of $\partial^1([s]_{\mathscr{F}})$ under the injection $C_*([s]_{\mathscr{F}}) \to C_*(T_s)$ induced by the (surjective) exponential map $T_s \to [s]_{\mathscr{F}}$. Then K_s is a compact subset of $C_*(T_s)$ for all $s \in S$.

By volume comparison with Euclidean space, every Riemannian manifold of nonnegative Ricci curvature has polynomial growth. It follows that $\mathscr{F}$ is amenable and, consequently, that $\mathscr{F}|S$ is as well. By [6], there exists a $\mathbf{Z}$-action on S whose orbits are a.e. the equivalence classes of $\mathscr{F}|S$.

The tangent bundle of $\mathscr{F}$ is measurably trivial, so, by the von Neumann selection theorem, we may choose a measurable system $\mu_s \in M(K_s)$. We may apply a standard averaging argument to the $\mathbf{Z}$-action and assume: for a.e. $(s, s') \in \mathscr{F}|S$, $\nu_s = \nu_{s'}$, where $\nu_s \in M(\partial^1[s]_{\mathscr{F}})$ is the preimage of μ_s under the bijection $\partial^1([s]_{\mathscr{F}}) \to K_s$.

For all $s \in S$, let $h_s \in C([s]_{\mathscr{F}})$ be defined by

$$h_s(l) := \int_{\partial^1[s]_{\mathscr{F}}} (h|_s^l)\, d\nu_s(h).$$

By Lemma 3.5, every element of every $\partial^1[s]_{\mathscr{F}}$ is Lipschitz-1 and weakly superharmonic. So $\{-h_s\}_{s\in S}$ is a compatible system of Lipschitz-1, weakly subharmonic functions. By Theorem 4.6, h_s is harmonic for a.e. $s \in S$.

A continuous convex combination of weakly superharmonic functions (e.g., any h_s) cannot be harmonic unless a.e. function is. So ν_s must be concentrated on harmonic function classes for a.e. $s \in S$. But, by Lemma 3.2, no element of any $\partial^1[s]_{\mathscr{F}}$ is even C^1, much less harmonic. So we arrive at a contradiction. q.e.d.

In the preceding proof, it was necessary to introduce the tangent bundle to $\mathscr{F}$ because that space has a natural measure-theoretic structure on it (see [11, p. 42, 1.11]). It would be interesting to know if there is some natural a priori way to put a measure-theoretic structure on the disjoint union $\bigcup_{s\in S}\partial^1[s]$; this would simplify and clarify the proof, as G. Stuck has pointed out.

Corollary 5.2. *Let $\mathcal{F}$ be a foliation of a compact manifold with a holonomy-invariant measure that is finite on compact subsets of transversals. Assume that almost every leaf (with respect to this measure) is a complete Riemannian manifold of nonnegative Ricci curvature. Then almost every leaf of $\mathcal{F}$ is the Riemannian product of a compact manifold and a flat Euclidean space.*

Proof. Under the assumptions of the corollary, the local product of a holonomy-invariant transverse measure and Riemannian volume along the leaves define a structure of foliated measure space with finite total measure (Definition 1.5).

Corollary 5.3. *Let N be a complete manifold of nonnegative Ricci curvature. Suppose N admits a properly discontinuous action by isometries of a discrete countable group Γ which has 'finite covolume' in the following sense: there exists an open subset $U\subset N$ with finite volume whose translates $\{\gamma U|\gamma\in\Gamma\}$ cover N. Then N splits isometrically as a product of a compact manifold and a flat Euclidean space.*

Proof. We follow [12, p. 1010]. Any countable discrete group Γ has a measure preserving action on a finite measure space (take $X_1=\{0,1\}^\Gamma$ with the product measure $\prod_\Gamma\mu$, where $\mu(0)=\mu(1)=1/2$, and let $(f\cdot\gamma)(\gamma')=f(\gamma'\gamma^{-1})$ for $\gamma\in\Gamma$, $f\in X_1$). This action is free on a conull invariant set if Γ is infinite (the case of finite Γ is trivial, since then N itself is of finite volume, hence compact); we discard this null set to get a finite measure space X on which Γ acts freely. Consider the diagonal action of Γ on $X\times N$:

$$\gamma(f,p)=(f\cdot\gamma,\gamma(p)).$$

The quotient space $(X\times N)/\Gamma=M$ has a natural structure of foliated measure space with finite total volume: to each $p\in U$ associate a neighborhood D_p, a finite subgroup Γ_p of Γ such that $\gamma D_p\cap D_p\neq\varnothing\Rightarrow\gamma\in\Gamma_p$, and a measurable subset T_p of X whose translates under Γ_p are disjoint and cover X. We may take a countable subcollection $T_i\times D_i$ of $\{T_p\times D_p\}_{p\in U}$ whose union has finite total (product) measure and whose translates under Γ are disjoint and cover $X\times N$. We may use the $T_i\times D_i$ as flow boxes in M since they map injectively under the quotient projection; the action of Γ on X being free, each leaf of the resulting foliation is isometric to N. The corollary now follows immediately from the main theorem.

6. Almost periodic metrics

Fix a positive integer d. Let $E_1, \cdots, E_d$ be the standard framing of the unit tangent bundle $T\mathbf{R}^d$. We will be using the theory of almost periodic functions and of the Bohr compactification (see [3]).

Definition 6.1. Let g be a metric on $\mathbf{R}^d$. Let $g_{ij} := g(E_i, E_j)$ for all i, j. We say that g is *almost periodic* if every g_{ij} is an almost periodic function on the locally compact abelian group $(\mathbf{R}^d, +)$. We say that g is *almost periodic to order* m if, for all $k = 1, \cdots, m$, every kth order partial of every g_{ij} (i.e., every $g_{ij,l_1\cdots l_k}$, in the usual notation) is almost periodic.

Let $\|\cdot\|$ denote the usual norm on $\mathbf{R}^d$. Let $e_1, \cdots, e_d$ denote the standard basis of $\mathbf{R}^d$.

Definition 6.2. For all $C > 0$, let B_C denote the collection of all bilinear forms $b\colon \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ such that

$$C^{-1}\|v\| \le b(v, v) \le C\|v\|$$

for all $v \in \mathbf{R}^d$. We identify B_C as a subset of $\mathbf{R}^{d\times d}$ by identifying b with the matrix $[b(e_i, e_j)]_{i,j=1,\cdots,d}$.

Definition 6.3. A Riemannian metric g on $\mathbf{R}^d$ is *quasi-flat* if, for some $C > 0$ and, all $p \in \mathbf{R}^d$, we have $g_p \in B_C$.

We are interested in studying metrics on $\mathbf{R}^d$ which are quasi-flat and almost periodic. Examples include metrics on the torus $\mathbf{T}^d$ lifted to $\mathbf{R}^d$. A broader class of examples arises as follows. Inject $\mathbf{R}^d \hookrightarrow \mathbf{T}^{d+1} := \mathbf{R}^{d+1}/\mathbf{Z}^{d+1}$ by some homomorphism. Fix $C > 0$ and choose a smooth map $\overline{g}\colon \mathbf{T}^{d+1} \to \mathbf{R}^{d\times d}$ such that $\overline{g}(\mathbf{T}^{d+1}) \subseteq B_C$. Then the composite $\mathbf{R}^d \hookrightarrow \mathbf{T}^{d+1} \to B_C$ defines a quasi-flat metric which is almost periodic to all orders. It is the content of Theorem 6.5 that there are not nearly so many examples in the presence of curvature assumptions.

Lemma 6.4. *Let g be a metric on $\mathbf{R}^d$ which is quasi-flat and almost periodic to order m. Then there exist*

(A) *a compact metrizable group $(A, +)$;*

(B) *an injective homomorphism $i\colon \mathbf{R}^d \to A$ with dense image;*

(C) *a constant $C > 0$; and*

(D) *a continuous function $\overline{g}\colon A \to \mathbf{R}^{d\times d}$;*

such that

(1) $\overline{g}(A) \subseteq B_C$;

(2) $g = \overline{g} \circ i$;

(3) *for any* $a \in A$, *the function*

$$p \mapsto \overline{g}(i(p) + a) \colon \mathbf{R}^d \to \mathbf{R}^{d\times d}$$

is m-times continuously differentiable.

Proof. Let $(A_1, +)$ denote the Bohr compactification of $(\mathbf{R}^d, +)$. That is, A_1 is a compact Abelian group containing a copy of $\mathbf{R}^d$, and every almost periodic function on $\mathbf{R}^d$ has unique extension to A_1 (see [3]). Except for the metrizability requirement on A in condition (A), we could set $A = A_1$.

Let $i_1 \colon \mathbf{R}^d \hookrightarrow A_1$ be the Bohr injection. Let $\overline{g}_1 \colon A_1 \to \mathbf{R}^{d\times d}$ be the unique extension of g. Then $g = \overline{g}_1 \circ i_1$ and, by continuity, $\overline{g}_1(A_1) \subseteq B_C$.

Let $C(A_1, \mathbf{R}^{d\times d})$ be the topological space of continuous functions $A_1 \to \mathbf{R}^{d\times d}$ with the sup-norm topology. Then A_1 acts on $C(A_1, \mathbf{R}^{d\times d})$ by $(a\cdot f)(b) = f(b-a)$. Let $S := \mathrm{Stab}_{A_1}(\overline{g})$, $A_2 := A_1/S$, and let $\pi \colon A_1 \to A_2$ be the natural homomorphism. Let $i_2 := \pi \circ i_1$. Except for the fact that i_2 may not be injective, we could set $A = A_2$. The map $\overline{g}_1$ factors through π, i.e., there exists a $\overline{g}_2 \colon A_2 \to \mathbf{R}^{d\times d}$ such that $\overline{g}_1 = \overline{g}_2 \circ \pi$.

Claim. A_2 is metrizable. Since A_2 is compact, it suffices to exhibit a continuous injection of A_2 into a metrizable topological space. The map $a \mapsto a \cdot \overline{g}_1 \colon A_1 \to C(A_1, \mathbf{R}^{d\times d})$ factors to a continuous injection $A_2 \hookrightarrow C(A_1, \mathbf{R}^{d\times d})$. Let $C(\mathbf{R}^d, \mathbf{R}^{d\times d})$ denote the space of all continuous functions $\mathbf{R}^d \to \mathbf{R}^{d\times d}$ endowed with the topology of uniform convergence on compact sets. Since i_1 has dense image, we have a continuous injection

$$f \mapsto f \circ i_1 \colon C(A_1, \mathbf{R}^{d\times d}) \hookrightarrow C(\mathbf{R}^d, \mathbf{R}^{d\times d}).$$

The composite $A_2 \hookrightarrow C(A_1, \mathbf{R}^{d\times d}) \hookrightarrow C(\mathbf{R}^d, \mathbf{R}^{d\times d})$ injects A_2 into a metrizable space, proving the claim.

Let $i_3 \colon \mathbf{R}^d \to \mathbf{T}^{d+1} := \mathbf{R}^{d+1}/\mathbf{Z}^{d+1}$ be some injective homomorphism, $i \colon \mathbf{R}^d \to A_2 \times \mathbf{T}^{d+1}$ be defined by $i(p) := (i_2(p), i_3(p))$, and A be the closure of $i(\mathbf{R}^d)$ in $A_2 \times \mathbf{T}^{d+1}$. Let $\mathrm{pr}_1 \colon A \to A_2$ denote projection onto the first coordinate. Define $\overline{g} := \overline{g}_2 \circ \mathrm{pr}_1$.

Conditions (1) and (2) in Lemma 6.4 follow from the construction of $\overline{g}$. Since i has dense image, condition (3) follows from the fact that g is almost periodic to order m.

Theorem 6.5. *Let* g *be a metric on* $\mathbf{R}^d$ *which is quasi-isometric to the usual flat metric and almost periodic to order two. Assume that either the* *sectional curvatures of* g *are all nonpositive or the Ricci curvatures of* g *are all nonnegative. Then* g *is flat.*

Proof. Let g, $(A, +)$, C, i, and $\overline{g}$ be as in Lemma 6.4. Both $\mathbf{R}^d$ and A are Abelian, hence unimodular. Consequently, the cosets of $i(\mathbf{R}^d)$ in A yield a foliation by smooth manifolds with transverse invariant measure μ such that: the Haar measure on A is equal to the integral of Haar along the cosets (i.e., leaves) against μ. Thus, the foliation has transverse invariant measure of finite total volume since A is compact.

For each $a \in A$, the differential of the map $p \mapsto i(p)+a: \mathbf{R}^d \to i(\mathbf{R}^d)+a$ transports the standard framing of $T\mathbf{R}^d$ to a framing of the tangent bundle of the leaf through a. This gives a well-defined global framing of the tangent bundle of the foliation.

The function $\overline{g}: A \to B_C$, together with this framing, defines a leafwise C^2 Riemannian metric which is (uniformly) leafwise quasi-flat, and hence complete. We now replace Haar measure along the cosets by the volume form coming from this metric. We replace Haar on A by the integral of these new leafwise measures against μ. (The transverse measure μ is not altered.) By uniform quasi-flatness, A remains a finite measure space and every open set still has positive measure. That is, conull sets are dense.

Let $R_{ijkl}: \mathbf{R}^d \to \mathbf{R}$ be the coordinates of the curvature tensor of g, and $\overline{R}_{ijkl}: A \to \mathbf{R}$ be the coordinates of the leafwise curvature with respect to the global framing of the tangent bundle to the foliation. Then $R_{ijkl} = \overline{R}_{ijkl} \circ i$. Every $\overline{R}_{ijkl}$ is continuous on A, so it suffices to show that $\overline{R}_{ijkl} = 0$ a.e. on A. That is, it suffices to show that a.e. leaf is flat.

If the sectional curvatures of g are nonpositive, then we are done by [12, Theorem 1, p. 1011].

If the Ricci curvatures of g are nonnegative, then, by Theorem 5.1, almost every leaf is isometric to a compact manifold cross a flat Euclidean space. However, every leaf is homeomorphic to $\mathbf{R}^d$ and thus has vanishing homology groups. Consequently, this compact factor must be a point and so a.e. leaf is flat.

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